Identification of Vibrating Flexible Structures

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This paper presents novel identification schemes to determine model parameters of vibrating structures. A time-domain identification method using transient response is discussed first. Next, a steady-state response method using nonresonant harmonic excitations is considered. An especially attractive method for uniquely identifying the parameters of a structure using both free and forced response is also discussed. Numerical results show that the methods are relatively immune to the presence of damping and many low-frequency modes with repeated or closely spaced frequencies.

Introduction

ACTIVE control of large space structures (LSS) necessitates a sufficiently accurate estimate of the parameters so that control laws can be tuned on-orbit to ensure stability and permit less control effort to be expended. Algorithms for design of insensitive or adaptive controls are not attractive due to the large number of degrees of freedom to be controlled. In fact, for most LSS application, the only feasible approach appears to be: 1) identify the structural parameters then 2) use this information to adjust the gains, and perhaps 3) use adaptive methods to change a small number of critical parameters in real time. This paper addresses issue 1 above.

Transient Response Identification Method

Many structural modal identification methods are available which extract modal characteristics, i.e., natural frequencies and mode shapes from a set of resonant steady-state responses due to a large number of harmonic excitations. These methods encounter analytical and numerical difficulties when the system frequencies are closely spaced and the "single mode resonant response" assumption is used. Also, the time required to achieve steady state may be prohibitively long for lightly damped, low-frequency structures. Time-domain techniques[1-4] for structural identification were first proposed by Ibrahim. Ibrahim's time-domain (ITD) method is a modal identification scheme. The ITD method has been successfully applied to reduce measurements from several laboratory experiments, however, this method has been found to lack reliable robustness. In some applications, rank deficient linear systems are encountered. Recently, Juang and Papa[5] have developed a more robust time-domain modal identification method; this method is based upon judicious use of singular value decomposition.

Identification in Configuration Space

Consider a vibrating structure governed by the following linear matrix differential equation:

$$M\ddot{x}+C\dot{x}+Kx=f \tag{1}$$

where x is the $n\times 1$ configuration vector of physical displacement, f the $n\times 1$ force vector, M the $n\times n$ symmetric positive definite mass matrix, C the $n\times n$ symmetric positive semidefinite damping matrix, and K the $n\times n$ symmetric positive semidefinite stiffness matrix. Dots denote differentiation with respect to time.

It is assumed that a LSS can be satisfactorily modeled in the form given by Eq. (1). Our objective herein is to identify the poorly known coefficient matrices M, C, and K or some parameterication therefore, e.g., the system eigenvalues and eigenvectors. Equation (1) can be rewritten as

$$[\ddot{x}^T(t)\ \dot{x}^T(t)\ x^T(t)]\begin{bmatrix} M \\ C \\ K \end{bmatrix}=f^T(t) \tag{2}$$

T denotes the matrix transpose operation.

Now, consider an idealized measurement process wherein the position, velocity acceleration, and forces are measured at discrete instants, say $t_1,t_2,\ldots,t_m$. Upon writing m measurement equations ($m\geq 3n$) identical to Eq. (2), one for each measurement time, the resulting matrix equations can be written as

$$ZP=U \tag{3}$$

where Z is an $m\times 3n$ coefficient matrix, whose j row contains measurements of the system response at time t_j:

$$j\text{th row of } Z=[\ddot{x}^T(t_j)\dot{x}^T(t_j)x^T(t_j)] \tag{4}$$

U is a $m\times n$ matrix containing the following forcing functions:

$$j\text{th row of } U=f^T(t_j) \tag{5}$$

$$P=[M\,|\,C\,|\,K]^T \tag{6}$$

P is a $3n\times n$ matrix containing the unknown mass, damping, and stiffness parameters.

Since the number of elements in each column of P is $3n$ and $m>3n$, Eq. (3) overdetermines the columns of P. The least-squares solution for P is given as

$$P=LU \tag{7}$$

where the least-squares operator is formally

$$L=(Z^TZ)^{-1}Z^T \tag{8}$$

For large systems, of course, the explicit inversion should be avoided in favor of the Q-R reduction, Cholesky decomposition, or a singular value decomposition approach for in-

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Received June 29, 1984; revision received Oct. 9, 1984.

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creased efficiency and robustness.[6] The computations subsequently summarized in this paper were done using the *Q-R* algorithm. The only theoretical requirement is that the *least-squares coefficient matrix Z have full rank* (*3n*). Physically, this rank condition can be achieved only if all degrees of freedom participate in the response. Hence, a fundamental requirement for identifying the structure is that the *excitation should have sufficient energy and frequency content to excite the higher modes of the system.* Qualitatively, it is also evident that the actuator locations and phase distribution of the actuator input are likely to be important. Thus, the mass, damping, and stiffness matrices of the structure can be identified, at least in principle. The method is obviously straightforward. However, it requires that the number of forces equal the order of the system. Also, acceleration, velocity, and displacement are to be measured at all of the degrees of freedom. These requirements pose obvious practical difficulties. It is shown in Ref. 7 that, in order to use a smaller number of forces than the number of degrees of freedom of the system, a priori knowledge of the mass matrix is required. It should be noted that the method, as presented above, does not require or exploit the symmetry of the system matrices and, hence, is applicable to a general dynamic system involving gyroscopic and circulatory forces.[7] It is also evident that the size of the linear systems which must be solved is $3n$. Therefore, unless the matrices do, in fact, possess special properties, it is anticipated that practical computational restrictions will require $n<50$ for this approach. Of course, the heavy redundancy implicit in M, C, and K as descriptions of distributed mass, damping, and stiffness often can be eliminated in terms of fewer physical parameters, but the estimation process then must be coupled with the structural modeling (e.g., finite element) process.

Identification in State Space

For control applications, the system dynamics is expressed in state equations. Introducing the "$2n$"-dimensional state vector

$$g(t) = [x^T(t)\ \dot{x}^T(t)]^T \tag{9}$$

Equation (1) can be written as

$$\dot{g} = Ag + Bf \tag{10}$$

where

$$A = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K & -M^{-1}C \end{array}\right] \tag{11}$$

is the plant dynamic matrix and

$$B = \left[\begin{array}{c} 0 \\ \hline D_1 \end{array}\right] \tag{12}$$

is the control distribution matrix. The structure of B is dependent upon type and location of the force inputs. When all degrees of freedom are not excited, the force vector contains zero entries and B will be a $(2n\times n_1)$ rectangular matrix, where n_1 is the number of excitations. The unknown parameters to be identified are the elements of matrices A and B.

Consider the lower partition of Eq. (10)

$$\ddot{x}(t) = -M^{-1}Kx(t) - M^{-1}C\dot{x}(t) + D_1 f(t) \tag{13}$$

Equation (13) can be written as

$$\ddot{x}^T(t) = [x^T(t)\ \dot{x}^T(t)\ f^T(t)] \begin{bmatrix} (-M^{-1}K)^T \\ (-M^{-1}C)^T \\ D_1^T \end{bmatrix} \tag{14}$$

The matrices $M^{-1}K$, $M^{-1}C$, and B_1 can be determined, following a procedure analogous to that outlined previously for configuration space identification. It is evident from Eq. (14) that the least-squares coefficient matrix includes the force vector. Immediately, it can be inferred that *the force vector should form an independent set for unique identification of the system parameters.* This statement holds true for configuration space identification also. It can be observed that $\dot{x}$ in the least-squares coefficient matrix, viz., Eq. (3), becomes independent of other variables only through the forcing functions.

Identification Using Orthogonal Polynominals

The measurement of acceleration, velocity, and displacement at all degrees of freedom, as required by the methods presented earlier, poses obvious practical difficulties. To obtain partial relief from this requirement, an orthogonal identification scheme is proposed. Orthogonal polynomials can be used to represent completely any function to a required degree of accuracy.[8]

Consider the lower portion of the state equations, viz., Eq. (13). Also, it is assumed that the accelerations and forces are measured at discrete instants of time. Then they can be expanded using orthogonal polynomials such as Chebyshev, Legendre, etc.

$$\ddot{x} = P_1 T(t) \tag{15}$$

where P_1 is a rectangular coefficient matrix and

$$T(t) = [T_0(t) T_1(t) \ldots T_{N-1}(t)]^T \tag{16}$$

is a column vector consisting of orthogonal polynomials. The intergral of $T_r(t)$ can be expressed via a recurrence relationship involving T_{r+1} and T_{r-1}. Integrating Eq. (15)

$$\dot{x} = P_2 T(t) + c' = y_1 + c' \tag{17}$$

Integrating further

$$x = P_3 T(t) + c't + c'' = y + c't + c'' \tag{18}$$

where P_1, P_2, and P_3 are $n\times N$ matrices containing the expansion coefficients. By substituting Eqs. (15), (17), and (18) into Eq. (13), the least-squares problem can be constructed as

$$[y^T(t)\ y^T(t)\ f^T(t)\ t\ 1] \begin{bmatrix} -(M^{-1}K)^T \\ -(M^{-1}C)^T \\ B_1^T \\ D_1^T \\ d_2^T \end{bmatrix} = \ddot{x}^T(t_j) \tag{19}$$

$$d_1 = -M^{-1}Kc' \tag{20}$$

$$d_2 = -M^{-1}Kc'' - M^{-1}Cc' \tag{21}$$

$M^{-1}K$, $M^{-1}C$, D_1, d_1, and d_2 can be estimated from Eq. (19). c' and c'' can be determined using Eqs. (20) and (21). Thus, the number of measurements are reduced by a factor of ⅔ and the initial displacement and velocity vectors, usually the equilibrium positions of the structure, are also estimated along with the parameters; it is also possible to use velocity or displacement measurements alone. Although several orthogonal polynomials exists, the use of Chebyshev polynomials have found a wide application[8] in solving linear and nonlinear differential equations. Since we are concerned with the inverse problem (i.e., given the response, the best estimate of the system's parameters is to be determined), Chebyshev polynomials seem to be the natural choice.

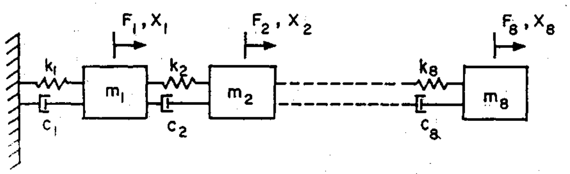
Numerical Results for Identification from Transient Free Response

Four specific linear systems are considered as representative examples; a spring-mass damper system (Fig. 1), a plane truss (Fig. 2), a cantilever beam (Fig. 3), and a rectangular membrane (Fig. 4) are considered to study the effects of 1) repeated low frequencies and rigid-body modes, 2) damping, 3) choice of excitation and number of excitations vs degrees of freedom, 4) excitation frequency vs system natural frequency, 5) measurement errors, measurement duration, and sampling interval, and 6) model truncation errors. Synthetic measured data are generated for each case using known parameter values. Table 1 gives the undamped eigenvalues of the examples. The proposed identification schemes performed very well for all four examples. The results are summarized below.

The plane truss example, as can be seen from Table 1, has three repeated eigenvalues and three zero eigenvalues corresponding to the rigid-body modes. Arbitrary viscous damping is included in the first and second example problems. It is found that the identification algorithms recovered the system matrices without any difficulty. It is clear that restrictive assumptions such as proportional or negligible damping are unnecessary for the success of the identification algorithms. Three excitation types were considered: 1) harmonic, 2) bang-bang (rectangular wave), and 3) frequency swept harmonic (harmonic excitations with time-varying frequency). Harmonic excitations yield good results for the spring-mass damper and cantilever beam. For the plane truss, bang-bang and frequency swept harmonic excitations are useful. Orthogonal polynomial identification of the plane truss with bang-bang excitation did not recover the parameters very well. The orthogonal polynomials are unable to represent the accelerations satisfactorily. However, with frequency swept harmonic excitation, the orthogonal identifier recovered all parameters accurately. Thus, the orthogonal identification scheme apparently works best with smooth and continuous excitations. The number of excitations required for the identification of spring-mass damper and cantilever beam can be as few as one. For the plane truss, a minimum of five excitations are needed.

Upon varying the frequency of excitation over the range of the system natural frequencies, no significant degradation in the performance of the algorithm is observed. Measurement errors introduce estimate errors, of course. The effect of discrepancies between commanded and realized excitations is also studied by including noise in the excitations.

The cantilever beam and a simply supported membrane are chosen to illustrate the effect of model truncation. These are distributed systems; we are interested in obtaining a discrete representation. The response of the cantilever beam is obtained by using the eigenfunctions and assuming that six modes participate in the response. Hence, a sixth-order model is obtained first. The model identified using either a bang-bang or harmonic excitation is the same. A reduced-order model (fourth order) is identified next. Table 2 compares the exact eigenvalues with those obtained from identified models.



$m_1 = m_2 = \ldots\ldots\ m_8 = m = 100$ Kg

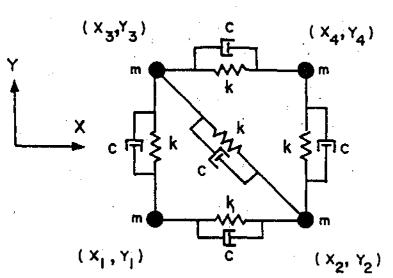
$k_1 = k_2 = \ldots\ldots\ k_8 = k = 36$ N/m

$c_1 = c_2 = \ldots\ldots\ c_8 = c = 45$ N.sec/m

$M = m\,[I],\ C = c\,[D],\ K = k[D]$

$$D = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ & & 2 & -1 & 0 & 0 & 0 & 0 \\ & & & 2 & -1 & 0 & 0 & 0 \\ & & & & 2 & -1 & 0 & 0 \\ & \text{sym} & & & & 2 & -1 & 0 \\ & & & & & & 2 & -1 \\ & & & & & & & 1 \end{bmatrix}$$

Fig. 1 Spring-mass damper.



$m = 100$ Kg, $k = 36$ N/m, $c = 45$ N.sec/m

$M = m\,[I],\ C = c\,[D],\ K = k[D]$

$$D = \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ & 2 & 0 & 0 & 0 & -2 & 0 & 0 \\ & & 3 & -1 & -1 & 1 & 0 & 0 \\ & & & 3 & 1 & -1 & 0 & -2 \\ & & & & 3 & -1 & -2 & 0 \\ & \text{sym} & & & & 3 & 0 & 0 \\ & & & & & & 2 & 0 \\ & & & & & & & 2 \end{bmatrix}$$

Fig. 2 Plane truss.

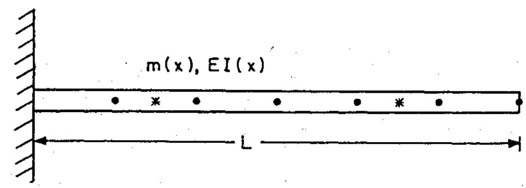


Fig. 3 Cantilever beam. $m(x)$ is the mass unit length, 2.4 kg/m; $EI(x)$ the bending stiffness, 495 N/m²; L the length, 3.0 m; * the position of sensors; and • position of actuators.

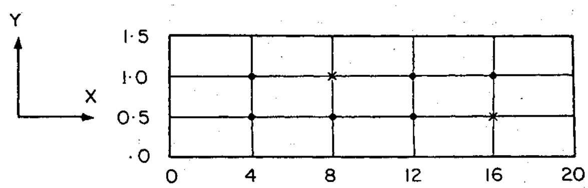


Fig. 4 Rectangular membrane. * is the position of sensors, and • the position of actuators.

Table 1 Eigenvalues of example problems

Spring-mass damper	Plane truss[a]	Cantilever beam	Rectangular membrane
0.1107	0.0	5.6105	21.0028
0.3284	0.0	35.1607	41.9173
0.5349	0.0	98.4510	21.1783
0.7232	0.4592	192.9246	42.0055
0.8868	0.8485	318.9182	21.4676
1.0203	0.8485	476.4086	42.1521
1.1190	0.8485	–	–
1.1796	1.1086	–	–

[a] Undamped eigenvalues.

It is also noted that the harmonic excitation resulted in a model that fits the measured acceleration fairly well. This is significant since accelerometers are the most commonly used sensors for vibration measurement. The bang-bang excitation, being rich in harmonic content, is able to excite the higher modes considerably and thus affects the identification of the reduced-order model. Table 3 gives the results for the membrane.

Identification Using Nonresonant Harmonic Excitations

The use of harmonic excitations for the identification of vibrating structures have received the attention of several investigators.[8-14] Raney[9] used such a scheme to successfully identify the effective masses, stiffnesses, and damping for a lightly damped structure having widely separated modes; the structures studied were 1/10- and 1/40-scale models of the Saturn launch vehicle. Several methods are suggested[9-14] to extract the normal modes from measured response. However, the use of normal modes is questionable when the damping in the system is not a proportional type. The methods using resonant harmonic excitations, as mentioned earlier, encounter both experimental and computational problems, when the system frequencies are closely spaced. A novel identification scheme using nonresonant harmonic excitations is presented in this section. The proposed scheme differs significantly from several of the existing methods which generally use resonant harmonic response to obtain the model parameters. The method requires that the structure be damped; the damping, however, can be arbitrary viscous damping.

Configuration Space Identification

Once again consider Eq. (1). Let the excitation $f(t)$ be given as

$$f_k(t) = \begin{Bmatrix} a_{1k}\sin(\omega_k t + \phi_{1k}) \\ a_{2k}\sin(\omega_k t + \phi_{2k}) \\ \text{-----------------} \\ \text{-----------------} \\ a_{nk}\sin(\omega_k t + \phi_{nk}) \end{Bmatrix}$$

Table 2 Comparison of eigenvalues for cantilever beam

	Eigenvalues obtained from		
		Fourth-order model	
Exact	Sixth-order model	Harmonic	Bang-bang
5.6105	5.6545	5.6282	11.2121
35.1607			
07	35.1664	35.2092	36.0591
98.451	98.4524	98.3340	98.3429
192.9246	192.9292	192.9283	192.9373
318.9182	318.8330	–	–
476.4086	475.9983	–	–

Table 3 Comparison of eigenvalues for rectangular membrane

	Eigenvalues obtained from		
		Fourth-order model	
Exact	Sixth-order model	Harmonic	Bang-bang
21.0028	21.0008	20.4976	21.2253
41.9173	41.9201	42.0707	41.9604
21.1783	21.0860	21.4256[a]	31.0774
42.0055	41.9602	–	–
21.4676	21.4405	21.4256[a]	34.5880
42.1521	42.1391	–	–

[a]Imaginary part given by eigenvalue routine is neglected.

or

$$f_k(t) = S_k\sin\omega_k t + C_k\cos\omega_k t \qquad k = 1,2,\ldots,m \tag{22}$$

S_k and C_k are the amplitudes of the sine and cosine components of the excitation. The steady-state response of the structure then can be written as

$$x(t) = A\sin\omega_k t + B_k\cos\omega_k t \qquad k = 1,2,\ldots,m \tag{23}$$

The structure is subjected to "m" harmonic excitations at frequencies $\omega_1, \omega_2, \ldots, \omega_m$. The excitation frequencies can be chosen arbitrarily and *need not* coincide with the system frequencies. For each excitation frequency "ω_k," the steady-state amplitudes A_k and B_k of the displacement are measured. Using Eqs. (22) and (23) in Eq. (2), we form the matrix equation

$$ZP = U \tag{24}$$

where P is the same as in Eq. (6).

$$k\text{th row of } Z = [-\omega_k^2 A_k^T \quad -\omega_k B_k^T \quad A_k^T] \tag{25}$$

$$(k+m)\text{th row of } Z = [-\omega_k^2 B_k^T \quad \omega_k A_k^T \quad B_k^T] \tag{26}$$

$$k\text{th row of } U = S_k^T \tag{27}$$

$$(k+m)\text{th row of } U = C_k^T \tag{28}$$

For $m > 3n/2$, Eq. (24) represents an overdetermined system of equations. The least-squares solution for P is given formally as

$$P = (Z^T Z)^{-1} Z^T U \tag{29}$$

Thus, system matrices M, C, and K can be identified directly from steady-state response. The same response data can be used to identify the system in state space. Also, the amplitudes of displacement must be measured at every degree of freedom. Alternatively, amplitudes of accelerations can be used. In that case, Eqs. (25) and (26) become

$$k\text{th row of } Z = [A_k^T \quad B_k^T/\omega_k \quad -A_k^T/\omega_k^2] \tag{30}$$

$$(k+m)\text{th row of } Z = [B_k^T \quad -A_k^T/\omega_k \quad -B_k^T/\omega_k^2] \tag{31}$$

A_k and B_k are the amplitudes of acceleration. It is assumed in the subsequent discussions that displacement amplitudes are measured and, hence, consider Eqs. (25) and (26) only.

State-Space Identification

Consider once again Eq. (13), which, using Eqs. (22) and (23), becomes

$$\begin{bmatrix} -\omega_1^2 A_1^T \\ \vdots \\ -\omega_m^2 A_m^T \\ \hline -\omega_1^2 B_1^T \\ \vdots \\ -\omega_m^2 B_m^T \end{bmatrix} = \begin{bmatrix} A_1^T & -\omega_1 B_1^T & S_1^T \\ \vdots & \vdots & \vdots \\ A_m^T & -\omega_m B_m^T & S_m^T \\ \hline B_1^T & \omega_1 A_1^T & C_1^T \\ \vdots & \vdots & \vdots \\ B_m^T & \omega_m A_m^T & C_m^T \end{bmatrix} \begin{bmatrix} (-M^{-1}K)^T \\ (-M^{-1}C)^T \\ D_1^T \end{bmatrix} \tag{32}$$

Equation (32) can be solved via least squares to obtain $M^{-1}K$, $M^{-1}C$, and D_1. It should be noted that the amplitudes of the excitations directly enter into the least-squares coefficient matrix. Hence, they should form an independent set. This is achieved by varying the phase of the excitation, i.e., ϕ_{1k}, $\phi_{2k}, \ldots, \phi_{nk}$ in Eq. (22) in a nonlinear fashion. This is precisely the reason for choosing the excitation of the form given by Eq. (22). The same requirement holds true for the configuration space identification, since the amplitudes of A_{1k} and B_{1k} can be linearly independent only via the excitation amplitudes.

Numerical Results for a Steady-State Response Identification

Two numerical examples, viz., a spring-mass damper (Fig. 1) and a plane truss (Fig. 2), are considered to study the effects of several implementation issues such as 1) closely spaced/repeated frequencies and rigid-body modes, 2) number of excitations, and 3) choice of excitation frequency. The results are summarized under each of the items (1-3).

The plane truss example is used to study the effects of repeated frequencies and rigid-body modes. Thirty excitation frequencies were used ranging from 0.6 to 0.89 rad/s in steps of 0.1 rad/s. All of the parameters are identified exactly in the absence of measurement noise. Thus, the proposed scheme is capable of identifying systems with closely spaced frequencies. For state-space identification, the number of excitations can be as low as three for the spring-mass damper system. However, the plane truss requires a minimum of four excitations.

The excitation frequency was varied within the range of the system's undamped natural frequencies. The authors considered those cases where excitation frequencies were below the lowest frequency of the structure and above the highest frequency of the structure. No significant degradation of the identification scheme was observed, thus validating the fact that the excitation frequencies can be chosen fairly arbitrarily.

Identification Using Free- and Forced-Response Data

Identification of mass and stiffness matrices from model test results has been reported by several authors. The objective of these identification schemes is to modify an a priori mass or stiffness matrix so that measured eigenvalues and eigenvectors agree with those of the analytical model. Berman,[15-17] using a minimization procedure, developed a noniterative scheme based on the orthogonality relationships of eigenvectors for computing a "nearest neighbor" update of the mass matrix. Following Berman's approach, Wei[18] developed a related method for correcting the stiffness matrix. Chen and Wada[19] discuss an interactive system parameter refinement procedure, employing the Jacobian matrix (consisting of the derivatives of eigenvalues and eigenvectors with respect to system parameters). Recently, Chen et al.[20] applied a first-order matrix perturbation approach to identify the mass and stiffness matrices. Other related approaches can be found in Refs. 21 and 22.

Free-response data are used herein to estimate the eigenvalues and eigenvectors of the system. Using orthogonality conditions, the matrices equal to system matrices multiplied by unknown scale factors are determined initially. These scale factors are then uniquely estimated by subjecting the system to known forces and measuring the acceleration, velocity, and displacement at several locations. The approach presented herein embodies a fundamental advantage: *perfect measurements, lead, to within truncation errors and arithmetic errors, to the true system parameters.*

Berman's Method: A Summary

In the absence of damping Eq. (1) reduces to

$$M\ddot{x} + Kx = 0 \tag{33}$$

The orthgonality conditions of the system described by Eq. (33) are

$$E^T M E = I \tag{34a}$$

$$E^T K E = \mathrm{diag}(\omega_1^2 \omega_2^2, \ldots, \omega_n^2) \equiv [\omega^2] \tag{34b}$$

where

E: $(n \times n)$ modal matrix

I: $(n \times n)$ identity matrix

$\omega_1, \omega_2, \ldots, \omega_n$ are the natural frequencies. Note that in Eqs. (34) the eigenvectors are normalized with respect to the mass matrix so that

$$e_i^T M e_i = 1 \qquad i = 1,2,\ldots,n \tag{35}$$

e_i is the ith eigenvector (ith column of E). It will be assumed that the measured modal matrix is square.

Berman[16] assumes an analytical mass matrix M_A. The measured eigenvectors are normalized with respect to M_A so that

$$e_i^T M_A e_i = 1 \qquad i = 1,2,\ldots,n \tag{36}$$

Letting ΔM be the desired correction matrix, Berman minimizes the Euclidian norm

$$\epsilon = \| N^{-1} \Delta M N^{-1} \| \tag{37}$$

subject to the constraint equation

$$E^T M E = I - m_a \qquad M = M_A + \Delta M$$

where $m_a = E^T M_A E$ is a nondiagonal matrix having unity as diagonal elements. Choosing $N = M_A^{1/2}$ as the weight matrix, Berman obtains

$$\Delta M = M_A E m_a^{-1} \, (I - m_a) m_a^{-1} E^T M_A \tag{38}$$

It can be seen from Eq. (38) that ΔM is symmetrical and determined to satisfy the orthogonality relations. However, one can obtain different "ΔM"s depending upon the choice of M_A. *Also, the decision to minimize ϵ, while reasonable, is nevertheless arbitrary.*

It is evident that the resulting mass and stiffness matrices are not unique. Subsequently, this truth will be illustrated with a simple numerical example. Hence, it is concluded that in order to determine the system matrices uniquely, some more conditions in addition to the (necessary but *not* sufficient) orthogonality conditions must be satisfied. These additional conditions can be readily obtained from the equations of motion. The free-response data can be used to determine the eigenvalues and eigenvectors of the system. The estimated modal data then can be used in conjunction with the *forced*-response data to uniquely identify the system matrices.

Identification of Eigenvalues and Eigenvectors: Rajaram and Junkins' Approach

A system described by Eq. (33) will be considered. The modal coordinate transformation is introduced as

$$x(t) = E\eta(t) \tag{39}$$

where $\eta(t)$ is the normal or modal coordinates of the system. Introducing Eq. (39) into the equations of motion, Eq. (33),

$$ME\ddot{\eta}(t) + KE\eta(t) = 0 \tag{40}$$

Multiplying Eq. (40) by E^T, we get

$$E^T ME\ddot{\eta}(t) + E^T KE\eta(t) = 0 \tag{41}$$

Due to the orthogonality properties of the eigenvectors, Eq. (41) represents a set of "n" uncoupled second-order equations. It the eigenvectors are normalized with respect to the mass matrix as per Eqs. (34), Eq. (41) becomes

$$\ddot{\eta}(t) + [\omega^2]\eta(t) = 0, \qquad [\omega^2] = \mathrm{diag}(\omega_1^2,\ldots,\omega_n^2) \tag{42}$$

When the eigenvectors are not normalized, Eq. (41) can be written as

$$M_m\ddot{\eta}(t) + K_m\eta(t) = 0 \tag{43}$$

M_m and K_m are diagonal matrices; the generalized "modal mass" and generalized "modal stiffness matrix," respectively. Also, K_m is related to M_m by the following relationship:

$$M_m = [\omega^2]K_m \tag{44}$$

The solution of Eq. (42) can be written as

$$\eta_i(t) = c_i\cos\omega_i t + s_i\sin\omega_i t \qquad i = 1,2,\ldots,n \tag{45}$$

c_i and s_i are constants depending upon $\eta_i(0)$ and $\dot{\eta}_i(0)$. Substituting Eq. (45) into the transformation Eq. (39), we obtain

$$x(t) = \sum_{i=1}^{n} (A_i\cos\omega_i t + B_i\sin\omega_i t) \tag{46}$$

where it is evident

$$A_i = c_i e_i \quad \text{and} \quad B_i = s_i e_i$$

Identifying either A_i or B_i is equivalent to identifying a scaled version of the ith normalized eigenvector. A Gauss-Newton least-squares differential correction method or a direct method based on the Fourier transform[23] of $x(t)$ can be used to obtain the modal parameters (ω_i, A_i, B_i).

We now turn attention to estimating the properly scaled mass and stiffness matrices. Equation (43), in the presence of forces, becomes

$$M_m\ddot{\eta}(t) + K_m\eta(t) = E^T f(t) \tag{47}$$

where the $(n\times 1)$ force vector $f(t)$ may contain zero entries, i.e., all of the degrees of freedom need not be forced. M_m and K_m are easily determined from the scalar components of Eq. (47), using the fact that $M_m(i,i) = \omega_i^2 K_m(i,i)$. It should be noted that, henceforth, the notation E will be used to represent the measured eigenvectors, normalized with respect to the a priori mass matrix. Since E is measured, transformation equation (39) can be used to transform measurements in physical space to modal space, i.e.,

$$\ddot{\eta}(t) = E^{-1}\ddot{x}(t) \tag{48a}$$

$$\eta(t) = E^{-1}x(t) \tag{48b}$$

Introducing Eqs. (48) into Eq. (47), for a known force vector, it is obviously possible to determine the diagonal elements of the modal stiffness matrix K_m and, using Eq. (44), M_m can be computed. The properly scaled, configuration space-mass matrix then can be obtained from

$$M = E^{-T}M_m E^{-1} \tag{49}$$

Similarly the stiffness matrix is given as

$$K = E^{-T}K_m E^{-1} \tag{50}$$

For high-dimensioned systems, of course, the inverses shown are replaced by appropriate matrix reduction algorithms. Thus, the parameter matrices can be estimated uniquely. We need to estimate only "n" parameters, viz., the diagonal elements of M_m. The elements of K_m can be derived from those of M_m through Eq. (44). Also, the amount of forced-response data required to estimate the modal matrices is not large. We need only "$2n$" measurements ("n" accelerations and "n" displacements), in addition to the measurement of forces.

Identification of Damped Systems

We now turn our attention to the necessary modifications of the preceding approach for including viscous (or equivalent viscous) damping. The equations of motion for a damped system are given by Eq. (1).

The eigenvalues and eigenvectors of a damped system are complex quantities. In order to apply classical modal analysis techniques, it is a usual practice to assume that the damping is either small or of a proportional type. Since the measured modes are complex, methods have been proposed to extract the normal modes from the complex modes. However, it is possible to rigorously apply a generalized modal analysis technique by transforming Eq. (1) from configuration to state space and estimate the system matrices, analogous to the previous section.

Introducing the state vector

$$g(t) = [x^T(t)\dot{x}(t)]^T$$

Equation (1) can be written as

$$M^*\dot{g}(t) + K^*g(t) = f^*(t) \tag{51}$$

where

$$M^* = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \tag{52a}$$

$$K^* = \begin{bmatrix} 0 & K \\ K & C \end{bmatrix} \tag{52b}$$

$$f^* = \begin{Bmatrix} 0 \\ f \end{Bmatrix} \tag{52c}$$

The eigenvalues and eigenvectors of a system described by Eq. (51) occur in complex conjugate pairs, i.e., if λ_i is an eigenvalue, $\bar{\lambda}_i$ is also an eigenvalue. Similarly e_i and $\bar{e}_i$ are the eigenvectors of the system. The orthogonality relations are

$$E^T M^* E = I \tag{53a}$$

$$E^T K^* E = -\Lambda \tag{53b}$$

where Λ is a diagonal matrix of the eigenvalues. Note that the modal matrix is of order $(2n\times 2n)$. Also, the eigenvectors are normalized with respect to M^* to satisfy

$$e_i^T M^* e_i = 1 \qquad i = 1,2,\ldots,2n \tag{54}$$

When the eigenvectors are not normalized, of course, Eqs. (53) become

$$E^T M^* E = M_m^* \tag{55a}$$

$$E^T K^* E = K_m^* \tag{55b}$$

where M_m^* and K_m^* are diagonal but complex matrices. The same notations as in the previous section are used. The eigenvectors have the form

$$e_i = \begin{bmatrix} a_i \\ \lambda_i a_i \end{bmatrix} \tag{56}$$

The free response of the system can be written as

$$x(t) = \sum_{i=1}^{n} (a_i e^{\lambda_i t} + \bar{a}_i e^{\bar{\lambda}_i t}) \tag{57}$$

where $\lambda_i (= -\alpha_i + j\omega_i)$ and the ith eigenvalue, and α_i is the damping factor and ω_i the damped frequency of oscillation. Equation (57) also can be written as

$$x(t) = 2\sum_{i=1}^{n} e^{-\alpha_i t}(C_i \cos\omega_i t - S_i \sin\omega_i t) \tag{58}$$

C_i and S_i are the real and imaginary components of a_i, respectively. A Gauss-Newton least-squares differential correction method[6] can be used to identify C_i, S_i, α_i, and ω_i $(i = 1,2,\ldots,n)$. Fast Fourier transform of $x(t)$ is quite useful in this case. The frequencies can be estimated from the power spectral density (psd) plot and used as a priori values in the Gauss-Newton algorithm. In this way, the convergence domain of the algorithm can be enhanced considerably. Using the orthogonality relations [Eqs. (56)] and the transformation

$$g(t) = \sum_{i=1}^{n} [e_i \eta_i(t) + \bar{e}_i \bar{\eta}_i(t)] = E\eta(t) \tag{59}$$

Eq. (52) reduces to

$$M_m^* \dot{\eta}(t) = K_m^* \eta(t) + E^T f^*(t) \tag{60}$$

where

$$\eta(t) = [\eta_1(t)\bar{\eta}_1(t)\ldots\eta_n(t)\bar{\eta}_n(t)]^T \tag{61}$$

is a complex modal coordinate vector. Equation (60) can be used to identify the M_m^* and K_m^* matrices from forced response, analogous to the previous section. After determining the modal matrices, M^* and K^* can be obtained from Eqs. (56). The method is similar to the one for the undamped system, except that the quantities involved are complex.

Numerical Examples Using Free and Forced Response

Example 1

Consider a two mass-spring system. The mass and stiffness are given as

$$M = \begin{bmatrix} 100 & 0 \\ 0 & 200 \end{bmatrix} \qquad K = \begin{bmatrix} 72 & -36 \\ -36 & 72 \end{bmatrix}$$

Choosing the a priori mass matrix M_A and stiffness matrix K_A for Berman's method[16,17] as

$$M_A = \begin{bmatrix} 90 & 0 \\ 0 & 220 \end{bmatrix} \qquad K_A = \begin{bmatrix} 65 & -32 \\ -32 & 79 \end{bmatrix}$$

The true system eigenvalues and eigenvectors are used as measurements. After carrying out algebra of Berman's method, the estimated mass matrix is found to be

$$M = \begin{bmatrix} 96.67 & 6.67 \\ 6.67 & 206.67 \end{bmatrix}$$

Although Berman's corrections to the diagonal elements are in the right direction, the off-diagonal elements' corrections are of comparable size and are no longer zero. A significantly *different* final mass matrix estimate would have been obtained, of course, if M_A were chosen differently. The estimated stiffness matrix for Berman's approach is found to be

$$K = \begin{bmatrix} 68.36 & -32.52 \\ -32.52 & 71.68 \end{bmatrix}$$

It can be noted that the diagonal terms are corrected fairly well while the corrections to off-diagonal terms are relatively small in this case.

Using the method developed above (with the eigenvalues and eigenvectors calculated using a finite Fourier transform), the mass matrix is determined from Eq. (49) to be

$$M = \begin{bmatrix} 100.00 & -0.8415E-04 \\ -0.8415E-04 & 200.0 \end{bmatrix}$$

and the estimated stiffness matrix, from Eq. (51), is calculated as

$$K = \begin{bmatrix} 71.995 & -35.997 \\ -35.997 & 71.9996 \end{bmatrix}$$

The small residual errors are the consequence of truncation of the Fourier transform of $x(t)$ to obtain eigenvalues or eigenvectors. If the Gauss-Newton iteration is used instead, the M and K matrix are recovered exactly (to eight digits). It is evident from this simple example that the proposed scheme correctly identifies the system matrices to within truncation errors in the finite Fourier transform. In essence, the scaling implicit within Berman's correction norm minimization is replaced by the requirement that the estimated M and K be consistent with a measured forced response.

Example 2

A two mass-spring damper system is considered. The various matrices are

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 5 & -4 \\ -4 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.2 \end{bmatrix}$$

The eigenvalues are

$$\lambda_1, \bar{\lambda}_1 = -0.222593 \pm j2.578255$$

$$\lambda_2, \bar{\lambda}_2 = -0.027406 \pm j0.545796$$

Using the Gauss-Newton method, the eigenvalues and eigenvectors are estimated from free response. The free-response data are properly scaled by applying an impulsive force on the second mass. The system matrices are obtained using the excitation $f_1 = 0.1$ sign (sin $0.2t$). The mass, stiffness, and damping matrices are identified exactly (eight digits). Thus the present method generalizes fully to include arbitrary viscous damping.

Conclusions

Three novel schemes are proposed to identify the parameters of vibrating structures. Numerical results on a variety of transparent examples support the validity of all three methods. The physical properties of mass, stiffness, and damping matrices are identified. All three proposed methods are applicable to damped structures. No assumptions regarding the nature of damping are made, other than it is of the viscous type. Systems with closely spaced frequencies present no apparent computational difficulty. In fact, example structures with repeated frequencies and rigid-body modes are identified reliably without difficulty. It is also shown that multiple excitation vectors should be chosen to form an independent set. The methods have been illustrated, however, only for low-dimensioned examples; significant future effort should consider high-degree-of-freedom systems to evaluate the robustness and relative merits of these approaches. It is also important to reduce the dimensionality by coupling the estimation algorithms to the structural modeling process.

Acknowledgments

This work was supported by the U.S. Air Force Office of Scientific Research under Contract F4920-83-K-0032. The technical liaison of Dr. A.K. Amos is most appreciated.

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